MATHEMATICS

SPHERICAL HARMONICS AND THE PRODUCT OF TWO JACOBI POLYNOMIALS

BY

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1. *Introduction and preliminaries*

In [3] BRAAKSMA and MEULENBELD proved the following Laplace representation of the Jacobi polynomial $P_n^{(\alpha,\beta)}(1-2t^2)$ for all $\alpha>-\frac{1}{2}$ and $\beta>-\frac{1}{2}$.

$$(1.1)\quad \left\{\begin{aligned} P_n^{(\alpha,\beta)}(1-2t^2) &= \frac{(-1)^n\,2^{2n}}{\pi(2n)!}\,\frac{\Gamma(n+\alpha+1)\ \Gamma(n+\beta+1)}{\Gamma(\alpha+\frac{1}{2})\ \Gamma(\beta+\frac{1}{2})}\cdot\\ &\int_{-1}^{1}\int_{-1}^{1}(tu\pm i\sqrt{1-t^2}\,v)^{2n}(1-u^2)^{\alpha-\frac{1}{2}}(1-v^2)^{\beta-\frac{1}{2}}\,du\,dv.\end{aligned}\right.$$

They arrived at the above representation after characterizing the Jacobi polynomials as spherical harmonics in q dimensions which are invariant for certain orthogonal transformations. It is our aim to prove in this note an integral representation for the product $P_n^{(\alpha,\beta)}(1-2s^2)P_n^{(\alpha,\beta)}(1-2t^2)$ using the same approach. We wish to thank Professor Braaksma for a simplification of our original proof, that was based on formula (2.10) and the completeness property of Jacobi polynomials.

Let $x=(x_1,x_2,\ldots,x_q)$ be an element of a Euclidean space E_q of q dimensions, let $(x,y)=x_1y_1+x_2y_2+\ldots+x_qy_q$ where $y\in E_q$ and let $|x|=\sqrt{(x,x)}$. Suppose that $q=q_1+q_2$ where q_1 and q_2 are positive integers, denote by R_{q_1} the subspace $\{x|x_{q_1+1}=x_{q_1+2}=\ldots=x_q=0\}$ of E_q and denote by R_{q_2} the orthogonal complement of R_{q_1} in E_q. Let $\Omega_q\subset E_q$, $\Omega_{q_1}\subset R_{q_1}$ and $\Omega_{q_2}\subset R_{q_2}$ be unitspheres with surface elements $d\omega_q$, $d\omega_{q_1}$ and $d\omega_{q_2}$ respectively. The elements of the unitspheres are denoted by the Greek letters ξ and η.

Let Δ_q be the Laplace operator in q dimensions. If $H_n(x)$ is a homogeneous polynomial of degree n in q dimensions which satisfies $\Delta_qH_n(x)=0$, then the function $S_n(\xi)=H_n(\xi)$ defined on Ω_q is called a spherical harmonic of degree n in q dimensions. For an introduction to spherical harmonics the reader is referred to ERDÉLYI [5], Ch. 11 and MÜLLER [6].

Let C be the class of all orthogonal transformations A which leave R_{q_1} invariant. Finally, we need the following theorem due to BRAAKSMA and MEULENBELD [3].

THEOREM (1.1). *Let $S_m(\xi)$ be a spherical harmonic of degree m on Ω_q*

such that $S_m(A\xi)=S_m(\xi)$ *for all* $A\in C$. *Write* $\xi=t\xi^1+\sqrt{1-t^2}\xi^2$, *where* $0\leqq t\leqq 1$ *and* $\xi^i\in\Omega_{q_i}$, $i=1, 2$. *Then*

$$S_m(\xi)=\begin{cases} 0 & \textit{if } m \textit{ is odd,}\\ \text{const. } P_n^{(\frac12 q_1-1,\,\frac12 q_2-1)}(1-2t^2) & \textit{if } m=2n.\end{cases}$$

2. *The product* $P_n^{(\alpha,\beta)}(1-2s^2)P_n^{(\alpha,\beta)}(1-2t^2)$

From now on q_1 will be an integer such that $2\leqq q_1\leqq q-2$. Let $C_n^\lambda(t)$ be a Gegenbauer polynomial and consider the function F defined on $[0, 1]\times\Omega_q$ by

$$F(s, \xi)=\int_{\Omega_{q_1}}\int_{\Omega_{q_2}} C_{2n}^{\frac12 q-1}((\xi, s\eta^1+\sqrt{1-s^2}\,\eta^2))\,d\omega_{q_1}(\eta^1)\,d\omega_{q_2}(\eta^2). \tag{2.1}$$

For each $A\in C$ we have that

$$\begin{aligned} F(s, A\xi) &= \int_{\Omega_{q_1}}\int_{\Omega_{q_2}} C_{2n}^{\frac12 q-1}((\xi, sA^{-1}\eta^1+\sqrt{1-s^2}\,A^{-1}\eta^2))\,d\omega_{q_1}(\eta^1)\,d\omega_{q_2}(\eta^2)\\ &= \int_{\Omega_{q_1}}\int_{\Omega_{q_2}} C_{2n}^{\frac12 q-1}((\xi, s\eta^1+\sqrt{1-s^2}\,\eta^2))\,d\omega_{q_1}(A\eta^1)\,d\omega_{q_2}(A\eta^2)\\ &= F(s, \xi).\end{aligned}$$

The last equality follows from the fact that $d\omega_{q_i}(A\eta^i)=d\omega_{q_i}(\eta^i)$ for η^i on Ω_{q_i}, $i=1, 2$. Furthermore, since $F(s, \xi)$ is a definite integral of spherical harmonics $C_{2n}^{\frac12 q-1}((\xi, \eta))$, it is a spherical harmonic of degree $2n$ itself. Thus, for fixed $s\in[0, 1]$, $F(s, \xi)$ satisfies the conditions of theorem (1.1). Using the notation of this theorem we conclude that

$$F(s, \xi)=k(s)\,P_n^{(\frac12 q_1-1,\,\frac12 q_2-1)}(1-2t^2). \tag{2.2}$$

If in formula (2.1) we choose $\xi=(t, 0, \ldots, 0, \sqrt{1-t^2})$, $0\leqq t\leqq 1$, write $F(s, \xi)= =f(s, t)$ and substitute $\eta_1{}^1=u$ and $\eta_q{}^2=v$, then we may rewrite formula (2.1) in the following form.

$$\begin{cases} f(s, t)=\text{const.}\\ \int_{-1}^{1}\int_{-}^{1} C_{2n}^{\frac12 q-1}(stu+\sqrt{1-s^2}\sqrt{1-t^2}v)\,(1-u^2)^{\frac12 q_1-3/2}(1-v^2)^{\frac12 q_2-3/2}\,du\,dv.\end{cases} \tag{2.3}$$

Since $f(s, t)=f(t, s)$, it follows from formula (2.2) with $\xi=(t, 0, \ldots, 0, \sqrt{1-t^2})$, $0\leqq t\leqq 1$, that

$$f(s, t)=\text{const. } P_n^{(\frac12 q_1-1,\,\frac12 q_2-1)}(1-2s^2)\,P_n^{\frac12 q_1-1,\,\frac12 q_2-1}(1-2t^2). \tag{2.4}$$

Combining formulas (2.3) and (2.4) and writing $\alpha=\frac12 q_1-1$ and $\beta=\frac12 q_2-1$, we have proved the following result.

THEOREM (2.1). *Let* α *and* β *be integers or half integers* $\geqq 0$. *Then*

$$\begin{cases} P_n^{(\alpha,\beta)}(1-2s^2)\,P_n^{(\alpha,\beta)}(1-2t^2)=\dfrac{\Gamma(\alpha+\beta+1)\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{\pi n!\,\Gamma(n+\alpha+\beta+1)\,\Gamma(\alpha+\frac12)\,\Gamma(\beta+\frac12)}\,\cdot\\ \qquad\int_{-1}^{1}\int_{-1}^{1} C_{2n}^{\alpha+\beta+1}(stu+\sqrt{1-s^2}\sqrt{1-t^2}\,v)\,(1-u^2)^{\alpha-\frac12}(1-v^2)^{\beta-\frac12}\,du\,dv.\end{cases} \tag{2.5}$$

(The constant factor in formula (2.5) can be verified by putting $s=0$ and $t=1$ in this formula).

Remark. Formula (2.5) also holds for arbitrary complex α and β, $\operatorname{Re}\alpha > -\frac{1}{2}$, $\operatorname{Re}\beta > -\frac{1}{2}$. This cannot be verified directly in a simple way. However, we may reason as follows. We may rewrite formula (2.5) as

$$(2.6)\quad \left\{\begin{aligned} &\binom{n+\alpha+\beta}{n} B(\alpha+\tfrac{1}{2}, \tfrac{1}{2})B(\beta+\tfrac{1}{2}, \tfrac{1}{2})\, P_n^{(\alpha,\beta)}(1-2s^2)\, P_n^{(\alpha,\beta)}(1-2t^2) = \\ &\qquad = \binom{n+\alpha}{n}\binom{n+\beta}{n} \cdot \\ &\qquad \int_{-1}^{1}\int_{-1}^{1} C_{2n}^{\alpha+\beta+1}(stu+\sqrt{1-s^2}\sqrt{1-t^2}\,v)\,(1-u^2)^{\alpha-\frac{1}{2}}(1-v^2)^{\beta-\frac{1}{2}}\,du\,dv \end{aligned}\right.$$

for $\alpha, \beta \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$.

From the explicit expressions for Gegenbauer and Jacobi polynomials may be deduced that $P_n^{(\alpha,\beta)}(t)$ is a polynomial of degree n in α and β and that $C_{2n}^{\lambda}(t)$ is a polynomial of degree $2n$ in λ. Therefore, for fixed n, s and t, the left and right hand sides of (2.6) are both analytic in α and β for $\operatorname{Re}\alpha > -\frac{1}{2}$ and $\operatorname{Re}\beta > -\frac{1}{2}$.

Now let $\operatorname{Re}\alpha \geqq \frac{1}{2}$ and $\operatorname{Re}\beta \geqq \frac{1}{2}$. Then we have the following estimates.

$$\left|\int_{-1}^{1}\int_{-1}^{1} C_{2n}^{\alpha+\beta+1}(stu+\sqrt{1-s^2}\sqrt{1-t^2}v)\,(1-u^2)^{\alpha-\frac{1}{2}}(1-v^2)^{\beta-\frac{1}{2}}\,du\,dv\right| \leqq$$
$$\leqq \text{const. } |\alpha|^{2n}|\beta|^{2n},$$

$$B(\alpha+\tfrac{1}{2}, \tfrac{1}{2}) = \int_0^1 t^{\alpha-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,dt = \sqrt{2}\int_0^{\pi/2} (\cos\theta)^{\alpha-\frac{1}{2}} \cos\tfrac{1}{2}\theta\, d\theta \leqq \text{const.},$$

and similarly $B(\beta+\frac{1}{2}, \frac{1}{2}) \leqq \text{const.}$

Furthermore $\binom{n+\alpha}{n}$, $\binom{n+\beta}{n}$ and $\binom{n+\alpha+\beta}{n}$ are polynomials of degree n in α and β. Therefore, for any fixed n, s and t there exists a positive constant k such that for $\operatorname{Re}\alpha \geqq \frac{1}{2}$, $\operatorname{Re}\beta \geqq \frac{1}{2}$ both sides of (2.6) are bounded in absolute value by $k|\alpha|^{3n}|\beta|^{3n}$. We now apply a theorem of Carlson, which states that if $f(z)$ is regular and of the form $O(e^{k|z|})$, where $k<\pi$, for $\operatorname{Re} z \geqq 0$, and if $f(z)=0$ for $z=0, 1, 2, \ldots$, then $f(z)$ is identically zero. See Titchmarsh [8], p. 186, theorem 5.81. Applying this theorem twice, once with respect to α and once with respect to β, we conclude that formula (2.6) is true for all α and β with $\operatorname{Re}\alpha \geqq \frac{1}{2}$ and $\operatorname{Re}\beta \geqq \frac{1}{2}$. By analytic continuation theorem (2.1) will be valid for $\operatorname{Re}\alpha > -\frac{1}{2}$ and $\operatorname{Re}\beta > -\frac{1}{2}$.

If $\alpha = \frac{1}{2}$ and $\beta = \lambda - \frac{1}{2}$ then the double integral in formula (2.5) can be reduced to a single integral.

$$P_n^{(\frac{1}{2},\lambda-\frac{1}{2})}(1-2s^2)\, P_n^{(\frac{1}{2},\lambda-\frac{1}{2})}(1-2t^2) =$$

$$(2.7)\quad \left\{\begin{aligned} &= \text{const.}\,\frac{1}{st} \\ &\int_{-1}^{1} [C_{2n+1}^{\lambda}(st+\sqrt{1-s^2}\sqrt{1-t^2}\,v) - C_{2n+1}^{\lambda}(-st+\sqrt{1-s^2}\sqrt{1-t^2}\,v)]\,(1-v^2)^{\lambda-1}\,dv \\ &= \text{const.}\,\frac{2}{st}\int_{-1}^{1} C_{2n+1}^{\lambda}(st+\sqrt{1-s^2}\sqrt{1-t^2}\,v)\,(1-v^2)^{\lambda-1}\,dv. \end{aligned}\right.$$

Here we have used the facts that $d/ds\ C_n^\lambda(s) = 2\lambda C_{n-1}^{\lambda+1}(s)$ and $C_n^\lambda(-s) = (-1)^n C_n^\lambda(s)$. See ERDÉLYI [5], p. 175 formula (16) and p. 176 formula (23).

Since

$$sP_n^{(\frac{1}{2},\lambda-\frac{1}{2})}(1-2s^2) = \frac{(-1)^n\ \Gamma(n+\frac{3}{2})\ \Gamma(\lambda)}{\sqrt{\pi}\ \Gamma(n+\lambda+1)}\ C_{2n+1}^\lambda(s)$$

(See ERDÉLYI [5], p. 176 formula (22)), we obtain from formula (2.7)

$$C_m^\lambda(s)\,C_m^\lambda(t) = \frac{\Gamma(m+2\lambda)}{2^{2\lambda-1}\,m!\,\Gamma(\lambda)\,\Gamma(\lambda)} \int_{-1}^{1} C_m^\lambda(st+\sqrt{1-s^2}\sqrt{1-t^2}\,v)\,(1-v^2)^{\lambda-1}\,dv \tag{2.8}$$

with $m=2n+1$. Formula (2.8) with $m=0, 1, \ldots$ is well-known. See ERDÉLYI [4], p. 177, formula (20).

If $q_1=1$ or $q_2=1$, suitable modifications in the reduction of the integral in formula (2.1) have to be made. Then this leads to formula (2.8) with $m=2n$. This last case may also be handled by taking the limit of formula (2.5) for $\alpha\downarrow -\frac{1}{2}$. Formula (2.5) may be rewritten as

$$\frac{P_n^{(\alpha,\beta)}(1-2s^2)}{P_n^{(\alpha,\beta)}(1)}\ \frac{P_n^{(\alpha,\beta)}(1-2t^2)}{P_n^{(\alpha,\beta)}(-1)} =$$

$$= \int_{-1}^{1}\left[\int_0^1 \frac{C_{2n}^{\alpha+\beta+1}(stu+\sqrt{1-s^2}\ \sqrt{1-t^2}\,v)}{C_{2n}^{\alpha+\beta+1}(0)}\ \frac{(1-u^2)^{\alpha-\frac{1}{2}}\,du}{\int_0^1(1-u^2)^{\alpha-\frac{1}{2}}du}\right]\ \frac{(1-v^2)^{\beta-\frac{1}{2}}\,dv}{\int_{-1}^1(1-v^2)^{\beta-\frac{1}{2}}\,dv}.$$

Letting $\alpha\downarrow -\frac{1}{2}$, we obtain

$$\frac{P_n^{(-\frac{1}{2},\beta)}(1-2s^2)}{P_n^{(-\frac{1}{2},\beta)}(1)}\ \frac{P_n^{(-\frac{1}{2},\beta)}(1-2t^2)}{P_n^{(-\frac{1}{2},\beta)}(-1)} =$$

$$= \int_{-1}^{1} \frac{C_{2n}^{\beta+\frac{1}{2}}(st+\sqrt{1-s^2}\sqrt{1-t^2}\,v)}{C_{2n}^{\beta+\frac{1}{2}}(0)}\ \frac{(1-v^2)^{\beta-\frac{1}{2}}\,dv}{\int_{-1}^1(1-v^2)^{\beta-\frac{1}{2}}\,dv}.$$

The left hand side of this identity is equal to

$$\frac{C_{2n}^{\beta+\frac{1}{2}}(s)}{C_{2n}^{\beta+\frac{1}{2}}(0)}\cdot\frac{C_{2n}^{\beta+\frac{1}{2}}(t)}{C_{2n}^{\beta+\frac{1}{2}}(1)}.$$

It may be remarked that formula (2.5) is not the real generalization of formula (2.8). In formula (2.5) the product of two Jacobi polynomials is expressed as a definite integral of a Gegenbauer polynomial. A proper generalization of formula (2.8) would be to express $P_n^{(\alpha,\beta)}(s)P_n^{(\alpha,\beta)}(t)$ as a definite integral of $P_n^{(\alpha,\beta)}$ depending on some (probably complicated) argument. Therefore formula (2.5) does not have the same applications to Jacobi polynomials as formula (2.8) has to Gegenbauer polynomials (cf. ASKEY [1], p. 47).

Putting in formula (2.5) $s=1$ we find an integral representation for Jacobi polynomials in terms of Gegenbauer polynomials.

THEOREM (2.2). *Let* $\operatorname{Re}\alpha > -\frac{1}{2}$ *and* $\operatorname{Re}\beta > -\frac{1}{2}$. *Then*

$$\begin{cases} P_n^{(\alpha,\beta)}(1-2t^2) = \\ = \dfrac{2(-1)^n}{\sqrt{\pi}}\ \dfrac{\Gamma(\alpha+\beta+1)\ \Gamma(n+\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\ \Gamma(n+\alpha+\beta+1)} \displaystyle\int_0^1 C_{2n}^{\alpha+\beta+1}(tu)\,(1-u^2)^{\alpha-\frac{1}{2}}\,du. \end{cases} \tag{2.9}$$

Formula (2.9) can be proved directly by using the Bateman integral

$$F(a,b;c+\mu;s)=\frac{\Gamma(c+\mu)}{\Gamma(c)\,\Gamma(\mu)}\int_0^1 t^{c-1}(1-t)^{\mu-1}\,F(a,b;c;st)\,dt,$$

where Re $c>0$, Re $\mu>0$ and $-1<s<1$, and the following hypergeometric representations

$$P_n^{(\alpha,\beta)}(1-2t^2)=\binom{n+\alpha}{n}F(-n,\,n+\alpha+\beta+1;\alpha+1;t^2).$$

$$C_{2n}^{\lambda}(t)=(-1)^n\frac{\Gamma(\lambda+n)}{n!\,\Gamma(\lambda)}F(-n,\,n+\lambda;\,\tfrac{1}{2};t^2).$$

See Erdélyi [4], p. 78 formula (2), Erdélyi [5], p. 170 formula (16) and p. 176 formula (21). For α and β integers or half integers $\geqq 0$ formula (2.9) can also be obtained by proving that the function ϕ_m defined on Ω_q by

$$\phi_m(\xi)=\int_{\Omega_{q_1}} C_m^{\frac{1}{2}q-1}((\xi,\eta))\,d\omega_{q_1}(\eta) \tag{2.10}$$

satisfies the conditions of theorem (1.1) and continuing in the same fashion as we did with the function F defined by formula (2.1).

In a slightly different way formula (2.9) also appears in Bavinck [2], p. 29. The special cases $\alpha=\frac{1}{2}$ and $\alpha\downarrow-\frac{1}{2}$ of formula (2.9) may be handled in a similar way as the special cases of formula (2.5). For $\alpha=\frac{1}{2}$ we obtain

$$\frac{tP_n^{(\frac{1}{2},\beta)}(1-2t^2)}{P_n^{(\frac{1}{2},\beta)}(-1)}=\frac{C_{2n+1}^{\beta+\frac{1}{2}}(t)}{C_{2n+1}^{\beta+\frac{1}{2}}(1)}$$

and for $\alpha\downarrow-\frac{1}{2}$

$$\frac{P_n^{(-\frac{1}{2},\beta)}(1-2t^2)}{P_n^{(-\frac{1}{2},\beta)}(1)}=\frac{C_{2n}^{\beta+\frac{1}{2}}(t)}{C_{2n}^{\beta+\frac{1}{2}}(0)}.$$

Therefore, formula (2.9) is a kind of generalization of the quadratic transformations for Gegenbauer polynomials.

Finally we remark that the Laplace representation for Jacobi polynomials as given in formula (1.1) also follows from theorem (2.1). For Jacobi and Gegenbauer polynomials the following limit formulas hold. See Szegö [7] formulas (4.21.6) and (4.7.9)

$$\lim_{s\to\infty} s^{-n}P_n^{(\alpha,\beta)}(s)=2^{-n}\binom{2n+\alpha+\beta}{n}$$

and

$$\lim_{s\to\infty} s^{-n}C_n^{\lambda}(s)=2^n\binom{n+\lambda-1}{n}.$$

If we divide both sides of formula (2.5) by s^{2n} and let $s \to \infty$, we obtain formula (1.1).

In a next paper one of the authors will prove an expansion of $C_{2n}^{\alpha+\beta+1}(stu+\sqrt{1-s^2}\sqrt{1-t^2}v$ as a double Gegenbauer series with respect to $C_k^{\alpha}(u)$ and $C_l^{\beta}(v)$. The first term of this expansion is contained in formula (2.5).

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